

MODEL INVESTIGATION

Different kinds of hydraulic structures such as dams, spillways, canal head works and diversion structures and hydraulic machines such as turbines and pumps are designed and constructed to yield efficiently the desired output. We have to ascertain that the designed structures after construction will definitely yield the desired output. In case, if the structure is found not to perform as per the design, then it is not possible to rectify the same as the structures are very massive. Therefore, it becomes imperative to study, in advance, how the structure or the machine would perform once it is constructed. For this purpose, one has to resort to experimental investigation. As the real structures or machines to be constructed in the field are very huge in size, it is not feasible to conduct these experimental investigations on the full size of the structure. Hence, a small scale replica of the actual structure is constructed and then tests are conducted to obtain the desired information. The small scale replica (imitation) of the actual structure or the machine is known as its *model* whiles the actual structure or machine is known as the *prototype*.

The model tests are quite economical and convenient as the design, construction and operation of the model may be altered several times if necessary, till all the discrepancies found in the model are eliminated and the most suitable design is obtained. On the basis of final results obtained from the model tests, the design of the prototype may be modified and also it may be possible to predict the behaviour of the prototype. However, the model test results can be used to obtain the performance of the prototype only if a complete similarity between the model and the prototype exists. This may be achieved as below.

TYPES OF SIMILARITIES

There are three types of similarities to be established for complete similarity to exist between the model and the prototype. They are:

1. Geometric similarity
2. Kinematic similarity
3. Dynamic similarity

1. Geometric similarity:

When a model and the corresponding prototype are said to be geometrically similar?

If the ratios of the corresponding length dimensions of the model and the prototype are equal, the model and the prototype are said to be geometrically similar. Such a ratio is called scale ratio.

For example, let the lengths, breadths and depths of a model and the corresponding prototype be respectively, L_m , b_m and d_m and L_p , b_p and d_p .

Then the length scale ratios are: L_m / L_p , b_m / b_p and d_m / d_p . If these scale ratios are equal, then the model and the prototype are said to be geometrically similar.

2. Kinematic similarity:

When a model and the corresponding prototype are said to be kinematically similar?

If the paths of the homologous moving particles are geometrically similar and if the ratios of the velocities as well as accelerations of the homologous particles are equal, kinematic similarity is said to exist between the model and the prototype.

What is a homologous point?

Consider a model and the corresponding prototype which possess geometric similarity. A point in the model and the corresponding point in the prototype are said to be homologous points.

Since, both velocity and acceleration are vector quantities (i.e., both have magnitude and direction), kinematic similarity implies that the directions of velocities and accelerations at corresponding points (i.e., homologous points) are parallel to each other and the ratios of magnitudes of both velocities and accelerations at corresponding points in the model and the prototype have constant values at all corresponding set of points. Some of the scale ratios which are useful in describing kinematic similarity are:

Time scale ratio, $T_r = T_m / T_p$

Velocity scale ratio, $V_r = V_m / V_p = (L_m / T_m) / (L_p / T_p)$
 $= (L_m / L_p) (T_p / T_m)$

$$= (L_m / L_p) \{1 / (T_m / T_p)\}$$

$$= L_r / T_r$$

3. Dynamic similarity:

When a model and the corresponding prototype are said to be dynamically similar?

If the ratios of all the forces acting at homologous points in the model and the prototype which possess both geometric and kinematic similarities are equal, then it is said that the model and the prototype possess dynamic similarity.

In the problems concerning fluid flow, the forces acting may be any one, or a combination of several of the following forces:

- (i) Inertia forces, F_i
- (ii) Friction or viscous forces, F_v
- (iii) Gravity forces, F_g
- (iv) Pressure forces, F_p
- (v) Elastic forces, F_e
- (vi) Surface tension forces, F_s

Inertia force, F_i , is the force of resistance offered by an inert mass to acceleration. According to Newton's law of motion, the magnitude of the inertial force is equal to the product of the particle mass and acceleration of the particle. The direction of the inertia force is opposite to the direction of the acceleration of the particle.

The conditions required for complete dynamic similarity are developed from the Newton's Second Law of Motion. In a flowing fluid, if a fluid particle of mass M is subjected to acceleration a , then the inertial force F_i of the particle equals ' Ma '. If all the above listed forces come into play in the fluid flow system under consideration, then the resultant force, $\sum F$, which is the vectorial sum of all the listed forces acting on the fluid particle, will be equal to the inertial force of the fluid particle, i.e.,

$$\sum F = F_v + F_g + F_p + F_e + F_s = Ma$$

For complete dynamic similarity to exist between the model and its prototype, the ratio of the inertia forces of the model and the prototype must be equal to the ratio of the resultant forces of the model and the prototype. i.e.,

$$(\sum F)_m / (\sum F)_p = (F_v + F_g + F_p + F_e + F_s)_m / (F_v + F_g + F_p + F_e + F_s)_p$$

$$= (Ma)_m / (Ma)_p \quad \dots\dots (1)$$

In addition to the above stated condition for complete dynamic similarity, the ratio of the inertia forces of the model and the prototype must also be equal to the ratios of the individual component forces of the model and the prototype. i.e.,

$$\begin{aligned} \text{(i) } (F_v)_m / (F_v)_p &= (Ma)_m / (Ma)_p \\ &\text{(or)} \\ (Ma/F_v)_m &= (Ma/F_v)_p \quad \dots\dots (2) \end{aligned}$$

$$\begin{aligned} \text{(ii) } (F_g)_m / (F_g)_p &= (Ma)_m / (Ma)_p \\ &\text{(or)} \\ (Ma/F_g)_m &= (Ma/F_g)_p \quad \dots\dots (3) \end{aligned}$$

$$\begin{aligned} \text{(iii) } (F_p)_m / (F_p)_p &= (Ma)_m / (Ma)_p \\ &\text{(or)} \\ (Ma/F_p)_m &= (Ma/F_p)_p \quad \dots\dots (4) \end{aligned}$$

$$\begin{aligned} \text{(iv) } (F_e)_m / (F_e)_p &= (Ma)_m / (Ma)_p \\ &\text{(or)} \\ (Ma/F_e)_m &= (Ma/F_e)_p \quad \dots\dots (5) \end{aligned}$$

$$\begin{aligned} \text{(v) } (F_s)_m / (F_s)_p &= (Ma)_m / (Ma)_p \\ &\text{(or)} \\ (Ma/F_s)_m &= (Ma/F_s)_p \quad \dots\dots (6) \end{aligned}$$

Thus, it may be mentioned that when both the model and the prototype are geometrically, kinematically and dynamically similar, then they are said to be completely similar or complete similitude exists between the two systems. However, the existence of dynamic similarity implies that both geometric and kinematic similarities exist between the model and the prototype. Hence, if dynamic similarity exists between the model and the prototype, they are said to be completely similar. Further, for complete similarity to exist between the model and the prototype, the dimensionless terms (or the π - terms) formed from the complete set of variables involved must be the same for both the model and the prototype.

DIMENSIONLESS NUMBERS (FORCE RATIOS)

When a mass is in motion, inertial force always exists. Hence, in order to develop the conditions for dynamic similarity, the ratio of inertial force and any one of the remaining forces listed previously is considered. Each of these ratios

will obviously be a non-dimensional factor. The various force ratios are discussed herein:

(a) Inertia force – viscous force ratio (Reynolds number)

We know that, Inertia force = mass x acceleration

Since, mass density $\rho = \text{mass} / \text{volume}$, mass can be expressed as the product of mass density ρ and volume. Acceleration is the rate of change of velocity. Hence, we have,

$$\begin{aligned} \text{Inertia force} &= (\text{mass density} \times \text{volume}) (\text{velocity} / \text{time}) \\ &= \text{mass density} \times (\text{volume} / \text{time}) \times \text{velocity} \end{aligned}$$

By definition, (volume / time) represents the discharge. Discharge is the product of cross-sectional area of flow, A and the velocity of flow, V , i.e., discharge, $Q = AV$

$$\text{So, Inertia force} = \rho (AV) V = \rho AV^2$$

As cross sectional area of flow passage, A , has dimensions of L^2 , we have,

$$\text{Inertia force, } F_i = \rho L^2 V^2$$

By definition, as per Newton's law of viscosity, we have, shear stress due to viscous force, F_v , is given by

$$\tau = \mu (dV / dy)$$

where, $\mu =$ coefficient of viscosity of fluid (or) simply, the dynamic viscosity of the fluid

$(dV / dy) =$ velocity gradient

$$\text{Viscous force, } F_v = \text{shear stress} \times \text{area} = \tau A = \mu (dV / dy) A$$

Assuming (dV / dy) to be linear, the above expression can be written as

$$F_v = \mu (V / y) A$$

As y represents the thickness of fluid film, it has dimensions of L . The dimension of the area ' A ' is ' L^2 '. Replacing ' y ' by ' L ' and ' A ' by ' L^2 ', the above expression for F_v becomes

$$F_v = \mu (V / L) L^2 = \mu VL$$

Now, the ratio between the inertia force, F_i , and the viscous force, F_v , is given by

$$(F_i / F_v) = (\rho L^2 V^2) / (\mu VL) = (\rho LV / \mu) = (VL / \nu)$$

where ν is the kinematic viscosity of the fluid.

The force ratio (or) non-dimensional ratio, $(\rho LV / \mu)$, is called the **Reynolds number**, Re or N_R .

The Reynolds number indicates the relative predominance of the inertia force to the viscous force occurring in the flow system. If the Reynolds number is larger, greater will be the relative magnitude of inertia force. If the Reynolds number is smaller, the greater will be the relative magnitude of viscous force.

(b) Inertia force – Gravity force ratio (Froude number)

From the previous discussion, we have, $F_i = \rho L^2 V^2$

As per Newton's second law of motion, force due to gravity can be expressed as

$$F_g = \text{mass} \times \text{acceleration due to gravity}$$

Mass can be expressed as the product of mass density, ρ and the volume; hence,

$$\begin{aligned} F_g &= (\text{mass density} \times \text{volume}) (\text{acceleration due to gravity}) \\ &= (\rho \times \text{volume}) \times g \end{aligned}$$

Volume has dimensions of L^3 . Replacing 'volume' by ' L^3 ', the above expression becomes

$$F_g = \rho L^3 g$$

Now, the ratio between the inertia force, F_i , and the gravity force, F_g , is given by

$$(F_i / F_g) = (\rho L^2 V^2) / (\rho L^3 g) = V^2 / Lg$$

The square root of this ratio, i.e., $(V^2 / Lg)^{1/2} = V / (Lg)^{1/2}$ is called the **Froude number**.

(c) Inertia Force – Pressure Force ratio (Euler number)

Pressure force, F_p can be expressed as the product of the pressure intensity, p and the area, A , over which it acts. i.e.,

$$F_p = p \times A$$

Area A has dimensions of L^2 ; Replacing 'A' by ' L^2 ', the above expression becomes

$$F_p = p \times L^2$$

Hence, the ratio between the inertia force, F_i , and the pressure force, F_p , is given by

$$F_i / F_p = (\rho L^2 V^2) / (p L^2) = \rho V^2 / p = V^2 / (p / \rho)$$

The square root of this ratio, i.e., $[V^2 / (p / \rho)]^{1/2} = [V / (p/\rho)^{1/2}]$ is called the **Euler number**, Eu or N_E . The reciprocal of Euler number, i.e., $[(p / \rho)^{1/2} / V]$ is sometimes known as 'Newton number'.

(d) Inertia force – Elasticity force ratio (Mach number)

Force due to elasticity, F_e , is expressed as the product of the bulk modulus of elasticity, K , of the flowing fluid and the area, A , over which the force acts, i.e.,

$$F_e = K \times A$$

As the dimensions of area, A , are L^2 , the above expression becomes

$$F_e = K \times L^2$$

The ratio between the inertia force, F_i , and the force due to elasticity, F_e , is given by

$$F_i / F_e = (\rho L^2 V^2) / (K L^2) = \rho V^2 / K = V^2 / (K / \rho) = V^2 / C^2$$

where, $C = (K / \rho)^{1/2}$ = velocity of sound in that fluid medium whose bulk modulus of elasticity, K , and mass density, ρ , are being considered.

The ratio (V^2 / C^2) is known as the '**Cauchy number**'. The square root of this ratio, i.e., (V / C) or $\{V / (K / \rho)^{1/2}\}$ is known as the '**Mach number**', Ma or N_M .

When $Ma > 1$, i.e., $V > C$, or in other words, the characteristic velocity of flow of the fluid is more than velocity of sound in that flow medium, the flow is said to be supersonic. When $Ma < 1$, i.e., $V < C$, or in other words, the characteristic velocity of flow of the fluid is less than velocity of sound in that flow medium, the flow is said to be subsonic. When $Ma = 1$, or $V = C$, the flow is considered to be sonic. When $Ma \gg 1$, i.e., $V \gg C$, then the flow is sometimes termed as hypersonic. A higher Mach number indicates the predominance of the effect of compressibility of the fluid. However, when the Mach number is relatively small, say, less than 0.4, the effect of compressibility of the fluid can be neglected.

(e) Inertia force – Surface tension force ratio (Weber number)

Force due to surface tension, $F_s = \sigma L$

where σ = surface tension of fluid in contact with, say, air (in N/m)

L = length of the fluid film over which the force due to surface tension acts

Hence, the ratio of the inertia force, F_i , and the surface tension force, F_s , is given by

$$F_i / F_s = (\rho L^2 V^2) / (\sigma L) = (\rho L V^2) / \sigma = [V^2 / \{ \sigma / (\rho L) \}]$$

The square root of this ratio, i.e., $[V / \{ \sigma / (\rho L) \}]^{1/2}$ is called the **Weber number**.

SIMILARITY LAWS OR MODEL LAWS

The results obtained from the model tests can be transferred to the prototype by the use of model laws. The model laws can be developed from the principles of dynamic similarity. The conditions for the existence of dynamic similarity between the model and the prototype are depicted by equations (1) to (6). In almost all hydraulic problems encountered in practice, for which model studies are required to be carried out, it is quite rare that all the forces, namely, F_i , F_g , F_v , F_p , F_e and F_s are simultaneously predominant in the flow phenomenon. Moreover, in most of the fluid flow problems, only one force in addition to the inertia force, F_i , is relatively more significant than the rest of the forces. The rest of the forces may either do not exist or may be of negligible magnitude. Under these circumstances, the various model laws have been developed depending upon the significant influence of each of the forces on the different fluid flow phenomena. In the derivation of these model laws, it has been assumed that for equal values of the dimensionless parameters the corresponding flow pattern in model and its prototype are similar.

(a) Reynolds Model Law

In case of flows where, in addition to the inertia force, the only other force of significance is the viscous force, the similarity in flow in the model and the prototype can be obtained if the Reynolds number of flow is the same in both the model and the prototype. This is known as **Reynolds Model Law**.

According to the law, we have,

$$(N_R)_{model} = (N_R)_{prototype}$$

$$(\rho_m V_m L_m) / \mu_m = (\rho_p V_p L_p) / \mu_p$$

where $(N_R)_{model}$ = Reynolds number of flow in model

$(N_R)_{prototype}$ = Reynolds number of flow in prototype

ρ_m = mass density of fluid in model

V_m = characteristic velocity of flow in model

L_m = characteristic length in model

μ_m = dynamic viscosity of fluid in model

ρ_p = mass density of fluid in prototype

V_p = characteristic velocity of flow in prototype

L_p = characteristic length in prototype

μ_p = dynamic viscosity of fluid in prototype

Dividing LHS by RHS of above equation,

$$\begin{aligned} [(\rho_m V_m L_m) / \mu_m] / [(\rho_p V_p L_p) / \mu_p] &= (\rho_m / \rho_p)(V_m / V_p)(L_m / L_p)(\mu_p / \mu_m) \\ &= \{(\rho_m / \rho_p)(V_m / V_p)(L_m / L_p)\} / \{(\mu_m / \mu_p)\} \\ &= \rho_r V_r L_r / \mu_r = 1 \quad \dots\dots (7) \end{aligned}$$

where ρ_r = Mass density scale ratio

V_r = characteristic velocity scale ratio

L_r = Length scale ratio

μ_r = dynamic viscosity scale ratio

Equation (7) may be used to obtain the scale ratios for various other physical quantities on the basis of Reynolds model law.

Some of the phenomena for which Reynolds model law can be a sufficient criterion for similarity of flow in the model and the prototype are:

- (i) flow of incompressible fluid in closed pipes

- (ii) motion of submarines completely under water
- (iii) motion of air planes
- (iv) flow around structures and other bodies immersed completely under moving fluids

(b) Froude Model Law

In case of flows where, in addition to the inertia force, the only other force of significance is the force of gravity, the similarity in flow in the model and the prototype can be obtained if the Froude number of flow is the same in both the model and the prototype. This is known as **Froude Model Law**.

According to the law, we have,

$$(Fr)_{model} = (Fr)_{prototype}$$

$$V_m / (L_m g_m)^{1/2} = V_p / (L_p g_p)^{1/2} \quad \dots\dots \quad (13)$$

- where $(Fr)_{model}$ = Froude number of flow in model
- $(Fr)_{prototype}$ = Froude number of flow in prototype
- V_m = velocity of flow in model
- L_m = characteristic dimension (length) in model
- g_m = acceleration due gravity at the site of model testing
- V_p = velocity of flow in prototype
- L_p = characteristic dimension (length) in prototype
- g_p = acceleration due gravity at the site of prototype

Dividing LHS by RHS of above equation,

$$\begin{aligned} V_m / (L_m g_m)^{1/2} / V_p / (L_p g_p)^{1/2} &= 1 \\ \rightarrow V_r / (g_r L_r)^{1/2} &= 1 \\ \rightarrow V_r &= (g_r L_r)^{1/2} \quad \dots\dots \quad (14) \end{aligned}$$

Since in most cases, as the value of g at the site of model testing will practically be the same as the value of g at the site of the proposed prototype, we have the scale ratio of g, i.e., $g_r = g_m / g_p = 1$

Hence, equation (14) becomes

$$\begin{aligned} V_r &= L_r^{1/2} \\ \rightarrow V_r / L_r^{1/2} &= 1 \quad \dots\dots \quad (15) \end{aligned}$$

Equation (14) or (15) may be used to obtain the scale ratios for various other physical quantities.

Some of the phenomena for which Reynolds model law can be a sufficient criterion for dynamic similarity of flow in the model and the prototype are:

- (i) Free-surface flows such as flow over spillways, sluices, etc.,
- (ii) Flow of jet from an orifice or nozzle
- (iii) Problems in which waves are likely to be formed on the surface
- (iv) Problems in which fluids of different densities flow over one another

(c) Euler Model Law

In case of fluid systems where, in addition to the inertia force, the only other force of significance is the force due to supplied pressures, the dynamic similarity in flow in the model and the prototype can be obtained if the Euler number of flow is the same in both the model and the prototype. This is known as **Euler Model Law**.

$$(Eu)_{model} = (Eu)_{prototype}$$

$$\left[\frac{V_m}{(p_m/\rho_m)^{1/2}} \right] = \left[\frac{V_p}{(p_p/\rho_p)^{1/2}} \right] \quad \dots \quad (20)$$

where $(Eu)_{model}$ = Euler number of flow in model

$(Eu)_{prototype}$ = Euler number of flow in prototype

V_m = velocity of flow in the model

p_m = intensity of fluid pressure in the model

ρ_m = mass density of fluid in the model

V_p = velocity of flow in the prototype

p_p = intensity of fluid pressure in the prototype

ρ_p = mass density of fluid in the prototype

Dividing LHS by RHS of above equation,

$$\left[\frac{V_m}{(p_m/\rho_m)^{1/2}} \right] / \left[\frac{V_p}{(p_p/\rho_p)^{1/2}} \right] = 1$$

$$\rightarrow \left[\frac{V_r}{(p_r/\rho_r)^{1/2}} \right] = 1 \quad \dots \quad (21)$$

Equation (21) represents the Euler Model Law which may be used to evaluate scale ratios for various other physical quantities.

Euler model law may be considered as an essential requirement for establishing dynamic similarity in an enclosed fluid system where the turbulence is fully

developed and the viscous forces are insignificant, and also the forces of gravity and surface tension are completely absent.

(d) Mach Model Law

In case of fluid flow phenomena where, in addition to the inertia force, the only other force of significance is the force resulting from elastic compression, the dynamic similarity in flow in the model and the prototype can be obtained if the Mach number of flow is the same in both the model and the prototype. This is known as **Mach Model Law**.

$$(Ma)_{model} = (Ma)_{prototype}$$

$$[V_m / (K_m / \rho_m)^{1/2}] = [V_p / (K_p / \rho_p)^{1/2}] \quad \dots \quad (22)$$

where $(Ma)_{model}$ = Mach number of flow in model

$(Ma)_{prototype}$ = Mach number of flow in prototype

V_m = velocity of flow in the model

K_m = bulk modulus of elasticity of fluid in the model

ρ_m = mass density of fluid in the model

V_p = velocity of flow in the prototype

K_p = bulk modulus of elasticity of fluid in the prototype

ρ_p = mass density of fluid in the prototype

Dividing LHS by RHS of above equation,

$$[V_m / (K_m / \rho_m)^{1/2}] / [V_p / (K_p / \rho_p)^{1/2}] = 1$$

$$\rightarrow [V_r / (K_r / \rho_r)^{1/2}] = 1 \quad \dots \quad (23)$$

Equation (23) represents the Mach Model Law which may be used to evaluate scale ratios for various other physical quantities.

The Mach model law finds extensive application in aerodynamic testing and in phenomena involving velocities exceeding the speed of sound. It is also applicable in hydraulic model testing for cases of unsteady flow, especially water hammer problems.

(e) Weber Model Law

In case of fluid flow phenomena where, in addition to the inertia force, the only other force of significance is the force resulting from surface tension, the dynamic similarity in flow in the model and the prototype can be obtained if the

Weber number of flow is the same in both the model and the prototype. This is known as **Weber Model Law**.

$$(We)_{model} = (We)_{prototype}$$

$$[V_m / \{ \sigma_m / (\rho_m L_m) \}^{1/2}] = [V_p / \{ \sigma_p / (\rho_p L_p) \}^{1/2}] \quad \dots\dots (24)$$

where $(We)_{model}$ = Weber number of flow in model
 $(We)_{prototype}$ = Weber number of flow in prototype
 V_m = velocity of flow in the model
 σ_m = surface tension of fluid in the model
 ρ_m = mass density of fluid in the model
 L_m = characteristic length in the model
 V_p = velocity of flow in the prototype
 σ_p = surface tension of fluid in the prototype
 ρ_p = mass density of fluid in the prototype
 L_p = characteristic length in the prototype

Dividing LHS by RHS of above equation,

$$[V_m / \{ \sigma_m / (\rho_m L_m) \}^{1/2}] / [V_p / \{ \sigma_p / (\rho_p L_p) \}^{1/2}] = 1$$

$$\rightarrow [V_r / \{ \sigma_r / (\rho_r L_r) \}^{1/2}] = 1 \quad \dots\dots (25)$$

Equation (25) represents the Weber Model Law which may be used to evaluate scale ratios for various other physical quantities.

Weber model law can be applied in the following cases:

- (i) flow over weirs involving very low heads
- (ii) very thin sheet of liquid flowing over a surface
- (iii) capillary waves in channels

TYPES OF MODELS

Hydraulic models can be broadly classified into two categories namely,

- (i) Undistorted Models
- (ii) Distorted Models

(i) Undistorted Models

An undistorted model is the one which is geometrically similar to its prototype, that is, the scale ratios for corresponding linear dimensions of the model and its prototype are same. As the basic condition of perfect similitude, i.e., geometric similarity, is satisfied, prediction in case of such models is relatively easy and many of the results obtained from the model tests can be transferred directly to the prototype.

(ii) Distorted Models

Distorted models are those in which one or more terms of the model are not identical with their counterparts in the prototype. As the basic condition of perfect similitude, i.e., geometric similarity, is not satisfied, the results obtained with the help of such models are liable to distortion and have more qualitative value only.

A distorted model may have either geometrical distortion, or distortion of hydraulic quantities or a combination of these.

What is geometric distortion?

The geometric distortion can either be *dimensional distortion* or *configurationally distortional*. For example, when the scale ratio adopted for the longitudinal dimension of the model and the prototype is different from the scale ratio adopted for the vertical dimension of the model and the prototype, the model is said to be *dimensionally distortional*. In general, when different scale ratios are adopted for the longitudinal, transverse and vertical dimensions, then it is said to be a distortion of dimensions.

Where dimensionally distorted models are frequently employed?

Distortion of dimensions is frequently adopted in river models where a different scale ratio is adopted for depth. In river models, the scales for vertical dimensions are larger than scales for horizontal dimensions. Such models are called '*vertically exaggerated models*'.

When the general configuration of the model does not bear a resemblance with its prototype, it results in a *configurationally distortional* model. For example, a river model will have a distortion of configuration if it is constructed with a bed-slope different from the one given by vertical exaggeration.

What is material distortion?

When the physical properties of the corresponding materials in the model and the prototype do not satisfy the similitude conditions, the material distortion arises.

Material distortion may have to be adopted in river models constructed for the studies of sediment transport.

Further, it may not be possible to obtain similitude in respect of certain uncontrollable hydraulic quantities such as time, discharge, etc., which may lead to distortion of hydraulic quantities.

Typical examples where distorted models are required:

- (i) Rivers
- (ii) Dams across very wide rivers
- (iii) Harbours
- (iv) Estuaries, etc.,

In all the above cases, the horizontal dimensions are large in proportion to the vertical ones.

What are the reasons for adopting distorted models?

- (i) to maintain accuracy in vertical measurements
- (ii) to maintain turbulent flow
- (iii) to obtain suitable bed material and its adequate movement
- (iv) to obtain suitable roughness condition
- (v) to accommodate the available facilities such as space, money, water supply and time.

Example 1: A spillway 7.2 m high and 150 m long discharges 2150 m³/s under a head of 4 m. If a 1:16 model of the spillway is to be constructed, find the model dimensions, head over the model and the model discharge.

Solution.

Prototype spillway:

Height of prototype spillway, $(Height)_p = 7.2 \text{ m}$

Length of prototype spillway, $L_p = 150 \text{ m}$

Discharge of prototype spillway, $Q_p = 2150 \text{ m}^3/\text{s}$

Head in prototype spillway, $(Head)_p = 4 \text{ m}$

Model spillway:

Length scale ratio, $L_r = L_m / L_p = 1 / 16$

Hence, length of model spillway, $L_m = (1 / 16) L_p = (1 / 16) (150) = 9.375 \text{ m}$

For geometric similarity of model and prototype,

$$\frac{(Height)_m}{(Height)_p} = \frac{L_m}{L_p}$$

$$\rightarrow (Height)_m = (Height)_p \frac{L_m}{L_p} = 7.2 \left(\frac{1}{16} \right) = 0.45 \text{ m}$$

$$\frac{(Head)_m}{(Head)_p} = \frac{L_m}{L_p}$$

$$(Head)_m = (Head)_p \frac{L_m}{L_p} = 4 \left(\frac{1}{16} \right) = 0.25 \text{ m}$$

The discharge equation for spillway is given by

$$Q = C_d L H^{3/2}$$

where C_d = coefficient of discharge of spillway

L = Length of spillway

H = Head of water in spillway

For kinematic similarity to exist between the model and the prototype, we have,

$$\frac{Q_m}{Q_p} = \frac{L_m H_m^{3/2}}{L_p H_p^{3/2}} = \frac{(9.375)(0.25)^{3/2}}{(150)(4)^{3/2}} = 9.765625 \times 10^{-4}$$

$$Q_m = (9.765625 \times 10^{-4}) Q_p = (9.765625 \times 10^{-4})(2150) = 2.1 \text{ m}^3/\text{s}$$

Example 2: In order to estimate the frictional head loss in a pipe 1 m in diameter, through which castor oil of specific gravity 0.96 and dynamic viscosity 9.9 *poise*, is to be transported at the rate of 5000 litres per second, a test was conducted on a pipe of diameter 50 mm using water at 15°C as the model fluid. Calculate the discharge required for the model pipe. If the head loss

in 40 m length of the model pipe is measured as 13.6 mm of water, determine the corresponding head loss in the prototype. Also obtain the value of Darcy's friction factor for the prototype. Given absolute viscosity of water at 15°C = 0.0131 poise.

Solution.

Prototype pipe:

Diameter of pipe, $D_p = 1 \text{ m}$

Cross-sectional area of pipe, $A_p = \frac{\pi}{4} D_p^2 = \frac{\pi}{4} (1)^2 = 0.7854 \text{ m}^2$

Specific gravity of castor oil transported through pipe = 0.96

Mass density of castor oil, $\rho_p = (\text{specific gravity of castor oil}) \times$
(mass density of water)
 $= 0.96 \times 1000 = 960 \text{ kg (mass) / m}^3$

Dynamic viscosity of castor oil, $\mu_p = 9.9 \text{ poise} = 9.9 \times 0.1 \text{ N.s/m}^2 = 0.99 \text{ N.s/m}^2$

Discharge rate of castor oil through the pipe, $Q_p = 5000 \text{ litres per second}$
 $= 5000 \times 10^{-3} \text{ m}^3 / \text{s}$
 $= 5 \text{ m}^3 / \text{s}$

Average velocity of flow through pipe, $V_p = Q_p / A_p = 5 / 0.7854 = 6.3662 \text{ m/s}$

Head loss due to friction in prototype pipe, $(H_f)_p = ?$

Darcy's friction factor for the prototype pipe, $f_p = ?$

Model pipe:

Diameter of pipe, $D_m = 50 \text{ mm} = 50 \times 10^{-3} \text{ m} = 0.05 \text{ m}$

Cross-sectional area of model pipe, $A_m = \frac{\pi}{4} D_m^2 = \frac{\pi}{4} (0.05)^2 = 1.963 \times 10^{-3} \text{ m}^2$

Mass density of water, $\rho_m = 1000 \text{ kg (mass) / m}^3$

Discharge of water required for the model pipe, $Q_m = ?$

Length of model pipe, $L_m = 40 \text{ m}$

Head loss due to friction in 40 m length of model pipe, $(H_f)_m = 13.6 \text{ mm}$
 $= 0.0136 \text{ m}$

Dynamic viscosity of water, $\mu_m = 0.0131 \text{ poise} = 0.0131 \times 0.1 \text{ N.s/m}^2$
 $= 0.00131 \text{ N.s/m}^2$

For dynamic similarity of model and prototype, Reynolds model law must be applicable.

Reynolds model law:

$$(N_R)_{\text{model}} = (N_R)_{\text{prototype}}$$

$$(\rho_m V_m L_m) / \mu_m = (\rho_p V_p L_p) / \mu_p$$

Here, the characteristic dimension of pipe is its diameter; hence, $L_m = D_m$ and $L_p = D_p$ in the above expression

$$\rightarrow \frac{(960)(V_m)(0.05)}{0.00131} = \frac{(1000)(6.3662)(1)}{0.99}$$

$$\rightarrow V_m = 0.1755 \text{ m/s}$$

Hence, discharge required for the model pipe, $Q_m = A_m \times V_m$

$$\begin{aligned} &= (1.963 \times 10^{-3}) \times (0.1755) \\ &= 3.446 \times 10^{-4} \text{ m}^3/\text{s} \\ &= 0.345 \times 10^{-3} \text{ m}^3/\text{s} \\ &= 0.345 \text{ litres per second} \end{aligned}$$

By dimensional analysis, the Resistance to flow is given by

$$R = \rho L^2 V^2 \phi \left(\frac{\rho V D}{\mu} \right)$$

As the Reynolds number for both the model and the prototype are the same, the value of the function ϕ in the above expression will be the same for both the model and the prototype. Hence, we can write,

$$\left(\frac{R}{\rho L^2 V^2} \right)_m = \left(\frac{R}{\rho L^2 V^2} \right)_p$$

Resistance to flow, $R = (\text{drop in pressure intensity due to head loss}) \times$
(cross-sectional area of pipe)

$$= (\rho g H_f) \left(\frac{\pi}{4} D^2 \right)$$

Hence, we have,

$$\left[\frac{(\rho g H_f) \left(\frac{\pi}{4} D^2 \right)}{\rho L^2 V^2} \right]_m = \left[\frac{(\rho g H_f) \left(\frac{\pi}{4} D^2 \right)}{\rho L^2 V^2} \right]_p$$

$$\rightarrow \frac{(H_f)_m}{(H_f)_p} = \left(\frac{D_p^2}{D_m^2} \right) \left(\frac{L_m^2}{L_p^2} \right) \left(\frac{V_m^2}{V_p^2} \right)$$

$$\text{As } \left(\frac{L_m}{L_p} \right) = \left(\frac{D_m}{D_p} \right) = \left(\frac{0.05}{1} \right)$$

$$\frac{(H_f)_m}{(H_f)_p} = \left(\frac{1}{0.05} \right)^2 \left(\frac{0.05}{1} \right)^2 \left(\frac{0.1755}{6.3662} \right)^2 = 7.5997 \times 10^{-4}$$

It is given that $(H_f)_m = 0.0136 \text{ m}$ of water in a length of 40 m

$$\begin{aligned} \text{Hence, } (H_f)_p &= (H_f)_m / (7.5997 \times 10^{-4}) = 0.0136 / (7.5997 \times 10^{-4}) \\ &= 17.896 \text{ m of castor oil} \end{aligned}$$

The corresponding length of prototype pipe in which this loss of 17.896 m occurs can be determined using the length scale ratio.

$$\frac{L_m}{L_p} = \frac{D_m}{D_p} = \frac{0.05}{1} = 0.05$$

$$\rightarrow L_p = L_m / 0.05 = 40 / 0.05 = 800 \text{ m}$$

$$\begin{aligned} \text{Hence, loss of head per metre length of prototype pipe} &= (H_f)_p / L_p \\ &= 17.896 / 800 \\ &= 0.02237 \text{ m of castor oil} \\ &= 22.37 \text{ mm of oil per } m \text{ length of pipe} \end{aligned}$$

Darcy-Weisbach equation for head loss due to friction in pipe is given by

$$H_f = \frac{fL V^2}{D 2g}$$

Applying the above equation for the model pipe, we have,

$$(H_f)_p = \frac{f_p L_p V_p^2}{D_p 2g_p}$$

$$\rightarrow 17.896 = \frac{f_p (800) (6.3662)^2}{1 (2)(9.81)}$$

$$\rightarrow f_p = \text{friction factor for the prototype pipe} = 0.01083$$